Self-phase modulation of spherical gravitational waves

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Self-phase modulation of spherical gravitational wave packets propagating in a flat space-time in the presence of a tenuous distribution of matter is considered. Analogies with respect to similar effects in nonlinear optics are explored. Self-phase modulation of waves emitted from a single source can eventually lead to an efficient energy dilution of the gravitational wave energy over an increasingly large spectral range. An explicit criterion for the occurrence of a significant spectral energy dilution is established.

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I. INTRODUCTION

It is well known that Einstein's equation describing gravitational waves is strongly nonlinear [1,2]. Nonlinear wave processes similar to those observed in optics can then eventually occur. Recently, the possible occurrence of self-phase modulation, harmonic generation and nonlinear wave mixing was considered [3,4].

This is an important issue in two different respects. First, from a theoretical point of view it is important to explore and understand the similarities and differences between gravitational wave phenomena and well known effects in nonlinear optics that have been tested in the laboratory. Second, and in more practical terms, it is important for the possible detection of gravitational waves. Detectors have been designed and built under the assumption that the frequency spectrum of gravitational waves emitted by astronomical objects is conserved, and that the wave intensities decrease as the inverse of the square of the distance. However, nonlinear processes can eventually lead to spectral energy dilution such that the energy density received within the detectable frequency bandwidth is significantly decreased. Other processes eventually contributing to energy dilution could be the coupling with plasma waves [5] and with photons [6,7].

For waves emitted from a single astronomical source, the main nonlinear effects that can occur are self-phase modulation (for short pulses, of the order of a few cycles) and harmonic cascades (for longer pulses). Attention was however called to the fact that, for parallel propagation, the strong nonlinearities associated with empty and flat space-time exactly cancel each other [4]. The existence of such a negative result is apparently due to the absence of gravitational wave dispersion. Only antiparallel wave configurations, which are not very relevant to waves emitted by single sources, can eventually survive [8].

In this work, we return to the problem of parallel wave interactions. We will focus on self-phase modulation of spherical waves emitted by isolated sources. For simplicity, we will consider a flat space-time, filled with a tenuous distribution of matter (interstellar medium, dust clouds, etc.). The matter distribution will guarantee the existence of wave dispersion, which is an important ingredient of self-phase modulation in nonlinear optics [9,10]. The nonlinear wave equation is established in Sec. II, and the dispersion properties of linear waves are discussed in Sec. III. Nonlinear evolution of a spherical gravitational wave packet is studied in Sec. IV, where a necessary criterion for the occurrence of a significant amount of self-phase modulation is established. Finally, in Sec. V we state the conclusions.

II. NONLINEAR WAVE EQUATION

We consider propagation of small amplitude gravitational waves, in a region of space-time where we have only a very tenuous distribution of matter, characteristic of the interstellar medium. We can then, in a first approximation, neglect the background field curvature, which is important only in a small region around the wave emitter and will not significantly influence the nonlinear process.

We are considering a flat space-time, perturbed by a small amplitude gravitational wave. This can be described by the metric tensor elements

$$g_{ij} = \eta_{ij} + h_{ij}, \tag{1}$$

where $|h_{ij}| \ll 1$ represent the gravitational wave, and η_{ij} are the metric tensor elements of flat space-time,

$$\eta_{00}=1, \qquad \eta_{ii}=-1 \quad (i=1,2,3), \qquad \eta_{ij}=0 \quad (i \neq j). \tag{2}$$

In this case, we can derive from Einstein's equation the following nonlinear wave equation:

$$\Box^2 h_{ik} = -2\kappa S_{ij} + 2R_{ik}^{(3)}, \tag{3}$$

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where $\kappa=(8\pi/c^2)G$, G is the gravitational constant, and $\Box^2$ is the d'Alembert operator in the usual form,

$$\Box^2=\partial^j\partial_j=\eta^{ij}\partial_i\partial_j, \tag{4}$$

where we use $\partial_i\equiv\partial/\partial x^i$. The nonlinear term $R_{ik}^{(3)}$ contains third order nonlinearities of the Ricci tensor. It can be explicitly written as

$$R_{ik}^{(3)}=-\frac{1}{4}\eta^{nm}h^{lp}[(\partial_j h_{mi}+\partial_i h^{ml}-\partial_m h_{il})(\partial_n h_{pk}+\partial_k h_{pn} -\partial_p h_{kn})-(\partial_k h_{pi}+\partial_i h_{pk}-\partial_p h_{ik})\partial_l h_{np}]. \tag{5}$$

Here we have neglected the second-order nonlinear term, $R_{ik}^{(2)}=0$, due to symmetry arguments [3]. In Eq. (3) we have also included the linear dispersion term associated with the matter distribution κS_{ij}, where S_{ij} is related to the energy-momentum tensor T_{ij} by

$$S_{ij}=T_{ij}-\frac{1}{2}\eta_{ij}T \tag{6}$$

where $T=T_i^i$ is the trace. Obviously, the present physical picture can be valid only if the matter distribution is so tenuous that its influence on space-time curvature can be neglected.

III. LINEAR DISPERSION RELATION

Let us first consider the properties of a linear wave, propagating radially from a given point source. In order to discuss the dispersion properties of this wave we first consider the linearized wave equation

$$\Box^2 h_{ik}=-\kappa S_{ij}, \tag{7}$$

and assume a plane wave solution of the form

$$h_{ij}=\epsilon_{ij}A\exp[iq_n x^n], \tag{8}$$

where $i=\sqrt{-1}$, A is the amplitude, ϵ_{ij} is a unit polarization tensor such that $\epsilon_{ij}^*\epsilon_{ij}=1$, and q_n are the components of the four-wave-vector. If the scale of variation of the amplitude A is much larger than the typical wavelength, we can use $\partial_j h_{ik}=iq_j h_{ik}$, and write the linear wave equation as

$$\eta^{jn}q_j q_n h_{ik}=2\kappa S_{ik}. \tag{9}$$

The perturbed energy-momentum tensor can be considered proportional to the local amplitude of the gravitational wave: $S_{ik}=w_{ik}A$, where the tensor w_{ik} depends on the properties of the medium. We are then led to the linear dispersion relation

$$\eta^{jn}q_j q_n=w, \tag{10}$$

where we have used $w=2\kappa\epsilon^{ik*}w_{ik}$. Particular examples of w can be found in the literature. For instance, the cases of a cold dust cloud [11] and of a magnetized plasma [12] are well established and do not need to be explicitly given here.

This dispersion relation is valid for any low amplitude gravitational wave propagating in flat space-time, in the presence of a tenuous matter distribution, and assuming the short wavelength approximation. In order to deal with spherical waves it is appropriate to use a spherical coordinate system (r,θ,ϕ), such that

$$x^0=ct,\quad x^1=r,\quad x^2=\theta,\quad x^3=\phi. \tag{11}$$

For waves propagating in the radial direction we have to replace Eq. (7) by the following wave solution:

$$h_{ik}=\epsilon_{ik}\frac{a}{r}\exp(iq_0x^0+iq_1x^1)=\epsilon_{ik}\frac{a}{r}\exp(iqr-i\Omega t), \tag{12}$$

where a is the new amplitude, $q=q_1$ and $\Omega=-q_0c$. Because the nondiagonal components η^{ik}, with $i\neq k$, are equal to zero, we can easily transform the dispersion relation (10) into

$$\left(\frac{\Omega^2}{c^2}-q^2\right)=w(r,\theta,\phi), \tag{13}$$

where we have retained the possibility of a non-uniform distribution of matter. We can see that the matter distribution can change the phase velocity of the gravitational wave, according to

$$v_f=\frac{\Omega}{c}=\sqrt{c^2+w}\simeq c+\frac{w}{2c^2}. \tag{14}$$

For the group velocity, we have

$$v=\frac{\partial\Omega}{\partial q}=\frac{qc^2}{\Omega}\left(1+\frac{1}{2q}\frac{\partial w}{\partial q}\right)=c\left(1+\frac{1}{2q}\frac{\partial w}{\partial q}\right)\sqrt{1-wc^2/\Omega^2}. \tag{15}$$

It is clear that, even if w is independent of q, the presence of a small amount of matter leads to a group velocity wave dispersion. It is known from nonlinear optics that wave dispersion is an essential ingredient of self-phase modulation [9]. It is the absence of dispersion that eventually explains the otherwise counter-intuitive result that self-phase modulation is absent for plane gravitational waves propagating in empty flat space-time [4]. The inclusion of matter is thus an essential ingredient of the present study.

IV. NONLINEAR WAVE PROPAGATION

We can now examine the possibility of a given radial wave, satisfying the above linear dispersion relation, interacting with itself, due to the nonlinear contributions contained in the term $R_{ik}^{(3)}$. The nonlinear contributions of S_{ik} could equally be included, but for simplicity they are neglected here. The existence of nonlinear wave coupling implies that the wave amplitude a in Eq. (12) can no longer be a constant, and is replaced by a slowly varying function of r and t. This means that we now have

$$\partial_j h_{ik} = \left(i q_j + \frac{1}{a} \partial_j a \right) h_{ik}. \qquad (16)$$

So we can write

$$\Box^2 h_{ik} \simeq [-\eta^{jn} q_j q_n + i \eta^{jn} q_j \ln a] h_{ik}. \qquad (17)$$

Assuming that the above linear dispersion relation still holds, we can cancel the first of these terms with the linear contribution from S_{ik}. We can write the second term as

$$i \eta^{jn} q_j \partial_n = -iq \partial_r - i \frac{\Omega}{c^2} \partial_t = -iq \left(\partial_r + \frac{1}{v} \partial_t \right) \qquad (18)$$

where $v = c^2/v_f$ is the group velocity.

In order to establish the nonlinear equation for the slowly varying amplitude a we now use an approximate expression for $R_{ik}^{(3)}$, where only the terms oscillating at the frequency of the wave Ω are retained

$$R_{ik}^{(3)} \simeq \frac{1}{2} \left(\frac{\Omega^2}{c^2} - q^2 \right) \frac{|a|^2}{r^2} \frac{a}{r} \exp(iqr - i\Omega t). \qquad (19)$$

In deriving this expression we have added to the solution (12) its complex conjugate, in order to adequately describe a real wave packet. For a given line of sight between the source at $r \simeq 0$ and the eventual observer at a finite distance r, we can use Eq. (13) with fixed values of θ and ϕ. Substituting in the nonlinear wave equation (3) we obtain

$$iq \left(\partial_r + \frac{1}{v} \partial_t \right) a = w(r) \frac{|a|^2}{r^3} a. \qquad (20)$$

This nonlinear equation for the slow wave amplitude clearly shows that the nonlinear effects disappear in the absence of matter, $w(r) = 0$, as noticed previously [4]. Let us make a variable transformation from the pair (r,t) to (z,τ), where we define $z = r - vt$ and $\tau = t$. We have then $\partial_r = \partial_z$ and $\partial_t = \partial_\tau - v \partial_z$. Replacing this in the above equation, we get

$$\partial_\tau a = -i \frac{w(z,\tau)}{q} v \frac{|a|^2}{r^2(z,\tau)} a. \qquad (21)$$

This equation is satisfied by a solution of the from

$$a(z,\tau) = a(z) \exp[i\phi(z,\tau)] \qquad (22)$$

with the phase function determined by

$$\phi(z,\tau) = \phi_0 - \int^\tau \frac{w(z,\tau')}{q} v \frac{|a|^2}{r^2(z,\tau')} d\tau'. \qquad (23)$$

This solution represents a gravitational wave packet propagating spherically with a nearly constant envelope $a(z)$ and a variable nonlinear phase. The wave frequency shift $\Delta\Omega$ will be given by the derivative of this phase with respect to the time variable t [10]. Neglecting the small variation of the distance $r(z,\tau)$ and the matter distribution $w(z,\tau)$ inside the wave packet envelope, this means that $\Delta\Omega$ will be essentially due to the variation of the energy distribution $|a(z)|^2$ with respect to time. But this envelope is a function of only $z = r - vt$ and we can use

$$\partial_t |a(z)|^2 = -v \partial_z |a(z)|^2. \qquad (24)$$

We can then state that

$$\Delta\Omega(\tau) = -v \partial_z \phi(z,\tau). \qquad (25)$$

Noting that, for short wave packets, the variation of the matter dispersion term $w(z,\tau)$ and distance with respect to the source $r(z,\tau)$ are negligible; we replace them in the expression of the phase by their central values $w(\tau) = w(z = 0,\tau)$ and $r(\tau) = r(z = 0,\tau)$. This leads to the following expression of the frequency shift occurring inside the wave packet envelope:

$$\Delta\Omega(\tau) = \int^\tau \frac{w(\tau')}{q} \frac{v^2}{r^2(\tau')} \partial_z |a(z)|^2 d\tau'. \qquad (26)$$

Here we notice that the distance traveled by the wave packet can be written as $r(\tau) = \int^\tau v(\tau') d\tau' \simeq c\tau$. Neglecting the possible slow change in the shape of the envelope over distance, we can finally write the above expression as

$$\Delta\Omega(\tau) \simeq \frac{1}{q} \partial_z |a(z)|^2 \int_0^\tau \frac{w(\tau')}{\tau'^2} d\tau'. \qquad (27)$$

In order to understand the physical meaning of this result, let us consider the simple case of a uniform distribution of matter along the entire line of sight. We can use $w(\tau) \simeq w_0 = \text{const}$, and get for the frequency shift, after a distance $r \simeq c\tau$ traveled by the wave packet

$$\Delta\Omega(r) \simeq -\frac{c}{q} w_0 \partial_z |A(z)|^2 r, \qquad (28)$$

where we have used the local spherical wave amplitude $A(z) = a(z)/r$, observed at a distance r from the source. This linear dependence of the frequency shift with time, or with the traveled distance, was found previously for linear propagation [3] and is well known from nonlinear optics [9]. Here, however, the frequency shift is proportional to the square of the local amplitude, which means that this effect can be significant only if it occurs over short distances, not far away from the emitter. This feature is specific to spherical waves propagating in uniform media.

Another interesting case is that of a non-uniform matter distribution where the wave packet propagates across a succession of N localized clouds, at distances r_i (with $i = 1, \ldots, N$) from the source, and widths $\Delta r_i \ll r_i$. We can then transform Eq. (27) in

$$\Delta\Omega(r) \simeq -\frac{c}{q} \sum_i w_i \partial_z |A_i(z)|^2 \Delta r_i \qquad (29)$$

where the local envelope amplitudes are determined by $A_i(z)=a(z)/r_i$. Again, the strongest contribution to the total frequency shift will result from the clouds located nearest to the source, supposing that they all have similar matter densities.

Let us assume that the source emits a gravitational wave packet with n cycles. Its width will be $\delta z \simeq 2\pi n/q$, and the maximum frequency shift associated with the closest cloud $(i=1)$ will be of the order of

$$\Delta\Omega_{max} \simeq \frac{c}{q} w_1 \frac{|A_1(z=0)|^2}{\delta z} \Delta r_1 . \tag{30}$$

We can also write $w_1=(\Omega/c)^2\alpha$, where $\alpha \ll 1$ is a small dimensionless factor. Noting that $\Omega \simeq qc$, this allows us to establish the necessary condition for a large frequency shift, leading to a significant spectral energy dilution, as $\Delta\Omega \geqslant \Omega$, or equivalently

$$\alpha |A_1|^2 \frac{\Delta r_1}{\delta z} = \frac{\alpha}{2\pi n} |A_1|^2 (q\Delta r_1) \geqslant 1. \tag{31}$$

Notice here that $(q\Delta r_1/2\pi)$ is the number of wavelengths over the cloud width. This can be a very large number. For instance, consider gravitational radiation coming from a stellar mass black hole with mass $M=10M_\odot$. The characteristic frequency of such black holes (the so called quasinormal frequencies) is ~ 1 kHz [13]. Consider now the situation in which the width is $\Delta r_1 = 10^{-3}$ pc, a dense enough cloud $(\alpha \sim 10^{-3})$, a short pulse $(n<10)$ and the optimal scenario of a close enough source $(|A_1| \sim 10^{-4})$. We would get $\Delta\Omega \geqslant \Omega \sim 10^{-5}$. This is a very small number, and it seems to rule out, for the moment, the immediate interest of such phenomena. However, the effect is there, and the present work confirms, on more solid grounds, the suggestion previously made [3] that self-phase modulation could eventually take place.

V. CONCLUSIONS

Nonlinear wave propagation of spherical gravitational waves was considered in this work. The possible occurrence of self-phase modulation was discussed. The case of short wave packets emitted from a point source in flat space-time was examined, where a tenuous distribution of matter was retained in order to guarantee linear wave dispersion, which is a necessary condition for self-phase modulation to occur. An explicit criterion for a significant spectral energy dilution due to self-phase modulation was established. It leads to the conclusion that the occurrence of self-phase modulation due to matter distribution very close to the gravitational wave source is plausible.

In contrast with what could occur with plane wave propagation, for spherical waves the contributions of phase modulation over distance decay very rapidly with distance from the source, due to wave amplitude decrease. For this reason, self-phase modulation is dominantly occurring very close to the emitter. It is important to note that our equations are not valid in the near zone of wave emission simply because the field is not weak there. Still, close to the source, *and* far away from the near zone, our formalism applies, and self-phase modulation may play an important role, depending on the strength of the source and the frequency of the waves.

Also notice that the region where this effect takes place is not necessarily the region where it can be observed, because the expanded wave spectrum will then propagate far away without further changes.

The efficiency of the self-phase modulation process is directly dependent on wave dispersion, which is a consequence of matter distribution. Curvature of space-time would also contribute to wave dispersion and would enhance the process. If, instead of spherical wave emission we have some kind of directionality, the wave amplitude decay will be smaller and phase modulation will also increase. Another source of nonlinearity is the energy-momentum tensor, or the matter distribution itself, which was not retained here. Space-time curvature, directionality effects and energy-momentum nonlinearities will eventually lead to more favorable criteria for the occurrence of self-phase modulation of gravitational waves, and will be considered in a future work.

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